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On Varisolvency and Alternation

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Recently, Barrar and Loeb [1] filled a gap in the theory of varisolvent families for the case that the degree of solvency is 1, 2, or 3. This partially answers the question whether the best approximation in the sense of Chebyshev alternates [2]. In this note we prove that in any family of varisolvent functions the best approximation alternates, if it has the maximal degree of solvency.

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Let X be a compact interval on the real line and let $C(X)$ be endowed with the topology induced by the Chebyshev norm. F is assumed to be a function unisolvent of variable degree [5]. Denote the degree of $F(a^*, x)$ by $m = m(a^*)$. In the definition of solvency, to each set of m distinct points $x_1, x_2, \ldots, x_m \in X$ a mapping from an open set of $\mathbf{R}^m$ into the family $F \subset C(X)$ is required, which associates to $(y_1, y_2, \ldots, y_m)$ an element $F(a, x)$ such that

$$F(a, x_i) = y_i, \qquad i = 1, 2, \ldots, m. \tag{1}$$

Referring to this mapping, we define the following:

DEFINITION 1. Let F be varisolvent. $F(a^*, x)$ is a normal element in F, if the defining mapping $(y_1, y_2, \ldots, y_m) \to F(a, x)$ is continuous for (at least) one set of distinct points $x_1, \ldots, x_m \in X$.

The degree of solvency is an upper semicontinuous function in F [5, Theorem 2]. Thus, it is constant in a neighborhood of a function with maximal degree. Therefore, in this case normality follows from Theorem 1.

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THEOREM 1. *Let F be a family unisolvent of variable degree. If $m(a)$ is continuous at $F(a^*, x)$, then $F(a^*, x)$ is a normal element in F.*

Proof. Let $F(a, x)$ be a solution of (1) with $m = m(a^*)$ and $|F(a^*, x_i) - y_i| < \delta$. For sufficiently small δ we have $m(a) = m(a^*)$ and Property Z implies uniqueness of the solution.[1] Now, it follows from solvency at $F(a, x)$ that the defining mapping for $F(a^*, x)$ is continuous at $y = (F(a, x_1), F(a, x_2),..., F(a, x_m))$.

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THEOREM 2. *If $F(a^*, x)$ is a normal element in a family of functions unisolvent of variable degree, then for every $\epsilon > 0$ there is an element $F(a, x)$ in F such that*

$$F(a, x) \geqslant F(a^*, x) \qquad \textit{for} \quad x \in X, \tag{2}$$

and $0 < \| F(a, \cdot) - F(a^*, \cdot)\| < \epsilon$.

Proof. Let $F(a^*, x)$ have degree $m = m(a^*)$. Assume that $x_1, x_2,..., x_m$ are chosen according to Definition 1. Let $I_1, I_2,..., I_m$ be open disjoint intervals such that the closure of $\bigcup_{k=1}^{m} I_k$ covers X, and $x_k \in I_k$, $k = 1, 2,..., m$. Consider the simplex in m-space

$$S = \left\{(z_1, z_2,..., z_m) \in \mathbf{R}^m; z_k \geqslant 0 \text{ for } k = 1, 2,..., m \text{ and } \sum_{k=1}^{m} z_k = 1\right\}. \tag{3}$$

Set $\delta = \frac{1}{2}\delta(a^*, \epsilon, x_1,..., x_m)$. By virtue of the solvency property for each $z \in S$ there exists a function in F with

$$F(\cdot, x_k) = F(a^*, x_k) + \delta \cdot z_k, \qquad k = 1, 2,..., m, \tag{4}$$

that, for convenience will be written as $F(z, x)$. We introduce m subsets

$$A_k = \{z \in S; \inf\{F(z, x) - F(\partial^*, x); x \in I_k\} \geqslant 0\}, \qquad k = 1, 2,..., m. \tag{5}$$

From the continuity of the mapping $z \to F(z, \cdot)$ it follows that the sets A_k are closed. Let d be the usual metric in m-space. Set

$$\rho_k(z) = \inf\{d(z, \zeta); \zeta \in A_k\}, \qquad k = 1, 2,..., m.$$

[1] Hobby and Rice [3] used another definition of solvency, which is not equivalent. In that case every element would be normal.

The proof of the theorem is complete, if we verify that

$$\bigcap_{k=1}^{m} A_k \neq \varnothing \tag{6}$$

holds. If (6) is not true, then we have for each $z \in S$:

$$\rho(z) = \sum_{k=1}^{m} \rho_k(z) > 0.$$

Consequently, the mapping

$$\psi: S \to S, \qquad \psi(z) = [1/\rho(z)](\rho_1(z), \rho_2(z), \ldots, \rho_m(z)) \tag{7}$$

is continuous. By virtue of Brouwer's fixed point theorem, there is a point $\bar{z} \in S$ satisfying $\psi(\bar{z}) = \bar{z}$. By the varisolvency property $F(\bar{z}, x) - F(a^*, x)$ has at most $m - 1$ zeros in X and there is no zero in at least one subinterval I_j. This implies $F(\bar{z}, x_j) \neq F(a^*, x_j)$ and

$$\bar{z}_j \neq 0.$$

On the other hand, the choice of I_j yields

$$\rho_j(\bar{z}) = 0,$$

contradicting $\bar{z} = \psi(\bar{z})$. Hence, the proof is completed.

We note that the subsets A_k satisfy the conditions of the theorem in [4, Section 2]. The equivalence of that theorem to Brouwer's fixed point theorem may be verified by considering the mapping ψ.

By the same arguments as in [1] we conclude from Theorem 2 that $F(a^*, x)$ is not a best approximation of $f(x) = F(a^*, x) + \frac{1}{2}\epsilon$. From this, we obtain the following:

COROLLARY 3. *If $F(a^*, x)$ is a normal element in F, then for every $\epsilon > 0$ there is a function $F(a, x)$ in F such that*

$$0 < F(a, x) - F(a^*, x) < \epsilon. \tag{8}$$

Since we have the strict inequality in (8), this improves Theorem 2.

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